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Five Dimensional Gauge Theories and Relativistic Integrable Systems

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We propose a non-perturbative solution of $N = 2$ supersymmetric gauge theory in five dimensions compactified on circle of a radius R. We consider the cases of the pure gauge theory as well as the theories with matter in the fundamental and in the adjoint representations. The pure theory as well as the one with adjoint hypermultiplet give rise to the known relativistic integrable systems with $\frac{1}{R}$ playing the rôle of the speed of light. The theory with adjoint hypermultiplet exhibits some interesting finiteness properties.

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1. Introduction

Recently there has been a tremendous progress in understanding the non-perturbative behavior of supersymmetric gauge theories in four dimensions. Namely, the exact low-energy effective action was determined for a large class of $N = 1$ and $N = 2$ theories. The main tool for such determination is the use of holomorphy of various functions and the electric-magnetic duality, constraining the geometry of the moduli space of scalars.

Of particular importance for our purposes is the solution of the pure $N = 2$ super-Yang-Mills theory originally obtained by Seiberg and Witten [1]and reinterpreted later by Gorsky et al. [2]in terms of integrable systems. The integrable system, relevant to the $N = 2$ gauge theory with the massive adjoint hypermultiplet was found by Donagi and Witten in [3].

In this paper we address the following question. Consider a five dimensional supersymmetric gauge theory. Take the space-time manifold to be $M = X \times \mathbf{S}^1_R$, where X is a four-manifold and R is the radius of the circle $\mathbf{S^1}$. One can think of the theory on M as of the theory on X with infinite number of massive fields, according to Kaluza-Klein ideology. It seems that one can integrate out the massive fields and define the effective action for the massless modes. The effective theory has four-dimensional $N = 2$ supersymmetry and the field content of the abelian minimal theory. Our goal is to determine its exact non-perturbative prepotential.

The low-energy four dimensional theory contains the photon(s) and therefore exhibits the usual electric-magnetic duality. This leads to the fact that the different regions of the moduli space of vacua are described by the different abelian theories, related to each other by duality transformations. This is a common feature of Coulomb branches of all $N = 2$ gauge theories.

The novelty of our problem in comparison with the conventional $N = 2$ pure Yang-Mills theory is the existence of a bigger group of discrete gauge transformations, which are present in the Coulomb phase. For example, in $SU(2)$ gauge theory in the $U(1)$ phase the scalar a related by $N = 2$ susy to the photon is acted on by the $\mathbb{Z}_2$ transformation: $a \to -a$ which is a remnant of the $SU(2)$ gauge invariance1. When the $N = 2$ theory is obtained by the compactification from five dimensions, the complex scalar ϕ comes from

1 for the general gauge group G the scalars in the low energy theory live in the Cartan subalgbera $\underline{\mathbf{t}}$ of the Lie algebra $\underline{\mathbf{g}}$ of G, and the residual gauge invariance is the action of the Weyl group W

the real scalar φ of the five dimensional theory and the fifth component A_t of the gauge field. Therefore, the residual gauge invariance includes the shifts $a \to a + \frac{in}{R}$, coming from the gauge transformations of the form $g(x,t) = \exp it\frac{n}{R}\sigma_3$[2], for $n \in \mathbb{Z}$. This implies that the gauge invariant order parameter

$$u \sim \mathrm{Tr}\phi^2 \sim a^2$$

of the pure Yang-Mills must be replaced by[3]

$$U \sim \mathrm{Tr}\exp 2\pi R\phi \sim \cosh\bigl(2\pi Ra\bigr)$$

(we are not careful with the numerical coefficients in these formulae). The low energy Lagrangian (more precisely, its part containing no more then two derivatives and four fermions) has the following form:

$$\mathcal{L} = \frac{1}{4\pi}\mathrm{Im}\left[\int d^4\theta \frac{\partial \mathcal{F}}{\partial A^i}\bar{A}^i + \frac{1}{2}\int d^2\theta \frac{\partial^2 \mathcal{F}}{\partial A^i \partial A^j} W^i_\alpha W^{\alpha,j}\right]$$

The coupling constants $\tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial A^i \partial A^j}$ non-trivially transform under the electric-magnetic duality group $Sp(2r,\mathbb{Z})$. In the rank one (for the gauge group $SU(2)$, say) case the effective gauge coupling constant

$$\tau(A) = \frac{\partial^2 \mathcal{F}}{\partial A^2} = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g^2}$$

transforms under electric-magnetic duality group $SL_2(\mathbb{Z})$ as

$$\tau \to \frac{m\tau + n}{r\tau + s}$$

One introduces a dual scalar a_D (which is an $N = 2$ superpartner of a dual photon): $a_D = \frac{\partial \mathcal{F}}{\partial a}$. The duality transformations mix a and a_D:

$$\begin{pmatrix} a \\ a_D \end{pmatrix} \to \begin{pmatrix} ra_D + sa \\ ma_D + na \end{pmatrix}$$

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2 There is a subtlety in identifying the periodicity of $g(x,t)$ in t, for in the theory with adjoint fields only one can assume that the gauge transformations are periodic up to the element of the center of the gauge group G only. This is discussed below.

3 for the general gauge group G the residual gauge symmetry acting on the scalars is the affine Weyl group and the order parameters are the characters of the group element $\exp(iR\phi)$ in the irreducible representations of G (there are $r = \mathrm{rk}G$ independent characters)

Stated more accurately, a and a_D appear as the central charges in the BPS representations of the $N = 2$ susy algebra. They depend holomorphically on the order parameter U (or u). This dependence comes both from the one-loop and instanton corrections.

Given the holomorphy and the duality properties of the solution we look for a holomorphic section $\big(a(U), a_D(U)\big)$ of an $SL_2(\mathbb{Z})$ bundle.

The solution of a pure Yang-Mills theory as presented in [1] makes use of a family of elliptic curves E_u:

$$i) \quad y^2 = (x^2 - \Lambda^4)(x - u)$$

or

$$ii) \quad z + \frac{\Lambda^4}{z} = p^2 - 2u$$

It has the form:

$$\begin{pmatrix} a(u) \\ a_D(u) \end{pmatrix} = \begin{pmatrix} \int_A \lambda \\ \int_B \lambda \end{pmatrix}$$

where λ is a meromorphic one-differential

$$i) \quad \lambda = \frac{ydx}{\Lambda^4 - x^2} \qquad \text{or} \qquad ii) \quad \lambda = p\frac{dz}{z}$$

Here Λ is a dynamically generated mass scale in the asymptotically free theory. This choice of a family of curves and the differential was interpreted in [2] in terms of an integrable system: make a change of variables:

$$y = \Lambda^2 p \sin\varphi, x = \Lambda^2 \cos\varphi, z = \Lambda^2 e^{i\varphi}$$

The family of curves E_u coincides with the family of the levels of a Hamiltonian of a periodic Toda chain:

$$u = \frac{p^2}{2} + \Lambda^2 \cos\varphi$$

while the differential λ is nothing but the canonical one-form $pd\varphi$. The choice of contours of integration - the cycles A and B is determined by the study of the limit $u \to \infty$. There $a \sim \sqrt{u}, a_D \sim a \log(u/\Lambda^2)$, as suggested by the perturbatively exact beta-function.

We will show that the solution of five dimensional theory can be obtained along the same lines with the periodic Toda chain replaced by its relativistic analogue [4]:

$$U = \frac{1}{R^2} \cosh Rp\sqrt{1 + 2(\Lambda R)^2 \cos\varphi}$$

For the $G = SU(N)$ case we obtain the corresponding family of curves, find the differential λ and compare it to the perturbative answer which we obtain with the help of a mathematical trick.

It is worth stressing that inspite of the fact that the additional states we get upon compactification are all massive with the masses $\sim \frac{1}{R}$ they do correct the metric on the space of vacua. The low-energy effective action has only low momenta modes but it describes the whole moduli space of vacua. As such, depending on the value of the scalar U the running of the effective coupling stops at the momentum (which runs inside the loop and so is not restricted by the external momentum being small), which can be very high in comparison to the external one.

The results of this paper extend the previous studies of the five dimensional gauge theory [5]. There, the perturbative prepotential for the five dimensional theory ($R = \infty$) was obtained by the perturbative computation in the heterotic string and compared with the predictions of the string duality (five dimensional supersymmetry arises in the compactifications of M-theory on a Calabi-Yau threefold). The prepotential in the compactifications of M-theory on CY is given by the intersection form of the CY (it is therefore piece-wise cubic):

$$\mathcal{F}_{5d} = \frac{1}{6} \sum_{i,j,k} C_{ijk} A^i A^j A^k \tag{1.1}$$

Upon the further compactification on a circle of the radius R the M-theory becomes the IIA string and one expects the classical intersection ring of the CY be replaced by its quantum cohomology which can be computed using mirror symmetry4. Our proposal provides a *local* description of these mirror CY's. The remark on the relation between the prepotential of the gauge theory in four dimensions and the quantum cohomology of CY doesn't explain the recently observed phenomenon that the prepotentials of the gauge theories obey the WDVV equations [6], [7]5.

2. Five dimensional theory

2.1. Perturbative prepotential

2.1.1.One-loop evaluation

The field content of a five dimensional supersymmetric gauge theory consists of a gauge field A_m, a real scalar φ and a gluino λ (all in the adjoint representation of the

4 I thank C.Vafa for pointing this out to me.

5 This has been explained to me by A. Morozov

gauge group G). We take the five-manifold M to be the product $M = X \times \mathbf{S}^1$ and denote the coordinates along X as x^μ, while the coordinate along $\mathbf{S}^1$ will be denoted as t.

The partition function on M in such a theory can be interpreted as a partition function in the infinite-dimensional supersymmetric quantum mechanics.

In order to see this explicitly let us analyze the supersymmetry generators. One of the generators6 acts as follows (we change the notation for the fermions - on a flat $X = \mathbb{R}^4$ this doesn't matter, on a general one this corresponds to the field content of a twisted theory. We also reintroduce an auxiliary field, a self-dual two -form $H^+_{\mu\nu}$) :

$$
\begin{aligned}
QA_\mu &= \psi_\mu & Q\psi_\mu &= F_{\mu t} + iD_\mu\varphi \\
Q(A_t + i\varphi) &= 0 & Q\chi^+_{\mu\nu} &= H^+_{\mu\nu} \\
Q(A_t - i\varphi) &= \eta & Q\eta &= D_t\varphi \\
QH^+_{\mu\nu} &= D_t\chi^+_{\mu\nu} + i[\varphi, \chi^+_{\mu\nu}]
\end{aligned}
\tag{2.1}
$$

This generator squares to the time translation together with the space-like gauge transformation (they can be both described as the result of the action of the operator $\nabla_t = \partial_t + A_t + i\varphi = D_t + i\varphi$). It has the meaning of the equivariant derivative with respect to the group of loops in the group of four-dimensional gauge transformations, extended by the circle, rotating the loops.

The action of the theory is a Q - commutator, apart from the "topological" term:

$$
S_{class} = \int_M \theta \wedge \mathrm{Tr} F \wedge F + \{Q, \int_M \chi F^+ - e_0^2 \chi H + \psi_\mu (F_{\mu t} - iD_t\varphi)\} \tag{2.2}
$$

with θ being a background abelian gauge field (it couples to the topological current $\mathrm{Tr}F \wedge F$). We expand all fields in the Fourier series in t and then use the localization argument, which reduces the path integral to the integral over the zero modes along t of the ratio of determinants one gets by expanding the action around the t-independent field configuration and treating the one-loop approximation.

In the low-energy limit the "would-be" potential $(D_t\varphi)^2$ vanishes and we can fix the gauge

$$
A_t + i\varphi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}
$$

6 Upon dimensional reduction down to four dimensions the theory becomes $N = 2$ supersymmetric theory and the generator we are about to consider corresponds to the scalar supercharge in the twisted version - topological theory

where a is t-independent. This leaves the freedom: $a \to -a$ and $a \to a + i\frac{n}{R}$, for $n \in \mathbb{Z}$.

Now let us expand the action around the time-independent configuration.

$$S = S_0 + S_2[A_n, \psi_n, \chi_n, \eta_n, \varphi_n]$$

The supersymmetry guarantees the exactness of the one-loop approximation. The result of the evaluation of the integral over the modes $A_n, \ldots$ together with the standard one-loop effective action of the pure four-dimensional theory looks like the standard low-energy action with the prepotential

$$\mathcal{F}^{\rm pert}_{5d}(a) = \frac{i}{8\pi^3 R^2} \sum_{\alpha} {\rm Li}_3\left(e^{2\pi R\langle \alpha, a\rangle}\right)$$

(Δ denotes the set of roots in $\underline{\mathbf{g}}$) which gives rise to the couplings:

$$\tau_\alpha = \frac{i}{2\pi} \log\left(\sinh^2\left(\pi R \langle \alpha, a\rangle\right)\right). \tag{2.3}$$

This is the "quantum deformation" of the standard perturbative prepotential of a pure $N = 2$ theory:

$$\mathcal{F}^{\rm pert}_{4d}(a) = \sum_{\alpha \in \Delta} \frac{i\langle a, \alpha\rangle^2}{8\pi} \log \frac{\langle a, \alpha\rangle^2}{\Lambda^2}$$

Indeed, in the limit $x \to 0$ the trilogarithm ${\rm Li}_3(e^x)$ goes over to $\frac{x^2}{4}\log x^2 + \ldots$.

To understand (2.3) it is instructive to consider the $G = SU(2)$ case where one has just one coupling $\tau(a)$ and rewrite it as:

$$\tau(a) = \frac{i}{2\pi} \log\left(\frac{a^2}{\Lambda^2}\right) + \sum_{n=1}^{\infty} \frac{i}{\pi} \log\left(a^2 + (n/R)^2\right) \tag{2.4}$$

One recognizes in the infinite sum the contribution of the excited Kaluza-Klein states (to avoid the confusion, this only takes into account the electrically charged states - the magnetically charged BPS strings are non-perturbative and appear only in the final expression). One also needs an ultra-violet regularization which gives rise to Λ.

<u>Decompactification limit.</u> In the opposite limit $x \to \infty$ one has ${\rm Li}_3(e^x) \sim -\frac{x^3}{6}+$ exponentially suppressed terms. This limit reproduces (1.1)for some specific choice of the coefficients C_{ijk}. In fact, in this limit all the non-perturbative corrections vanish (as the contribution of the n instantons is of the order of $\sim e^{-\frac{nR}{g_5^2}}$) and one is left with the purely

perturbative result. In fact, the large x asymptotic of $\mathrm{Li}_3(e^x)$ contains the quadratic as well as the linear (in x) terms which are irrelevant in the large R limit (one has to divide the four-dimensional expression for the prepotential by $2\pi R$ before taking the limit).

One can also notice the non-trivial chamber structure of the moduli space in the decompactified theory. It comes from the monodromy properties of the trilogarithm (see [8] for the discussions of the similar effects in the context of the heterotic string compactification). The answer for the five dimensional prepotential (the scalars a_i become real) looks like:

$$\mathcal{F}_{5d} = \frac{1}{6}\sum_{i,j} |a_i - a_j|^3 \tag{2.5}$$

There might be also a quadratic piece, corresponding to the finite bare coupling in five dimensions [9]. The non-smoothness of $\mathcal{F}_{5d}$ at the walls of the Weyl chamber has to do with the restoration of non-abelian gauge symmetry [10].

2.1.2.Instanton derivation

From the mathematical point of view the five dimensional theory is a way of study of the geometry of moduli space of instantons on X. More precisely, under some topological restrictions the moduli space $\mathcal{M}$ is a spinor Riemannian manifold and one may study the index of Dirac operator D, acting on spinors on $\mathcal{M}$.

The index of D is given by Atiyah-Singer theorem:

$$\mathrm{Ind}D = \int_{\mathcal{M}} \hat{A}(\mathcal{M}) = \int_{\mathcal{M}} \prod_i \frac{x_i/2}{\sinh(x_i/2)}$$

where the symmetric polynomials of the indeterminants x_i's are to be found from the Riemann-Roch-Grothendieck (or Atiyah-Singer family index) theorem:7

$$\sum_i e^{tx_i} = \int_X \hat{A}(X)\mathrm{Tr}_V \exp\left(\frac{t\phi + t^{1/2}\psi + F}{2\pi i}\right) \tag{2.6}$$

Here the fields ϕ, ψ, F are the scalar, the one-form fermion and the curvature of the gauge field in the twisted $N = 2$ theory (which is often reffered to as the Donaldson theory, as constructed by E.Witten [11]). The trace is taken in the adjoint representation V (the details are provided in [12]). Formally one can pass from the expression for

$$\sum_i x_i^n$$

7 In the general case the $\hat{A}(X)$ must be replaced by the Todd class $Td(X)$

to the $\hat{A}(\mathcal{M})$. One gets:

$$\hat{A}(\mathcal{M}) = \exp \int_X \left(\mathcal{O}^{(4)}_{\mathcal{F}} + (\alpha \mathrm{Tr} R^2 + \beta \mathrm{Tr} R\tilde{R}) \mathcal{O}^{(0)}_{\mathcal{P}} \right) \tag{2.7}$$

where $\mathcal{O}^{(i)}_{A}$ denotes the i-observable, constructed by means of the standard descend equations [11] out of the gauge invariant functional $A(\phi)$. The coefficients α and β in front of the densities $\mathrm{Tr} R \wedge R$ and $\mathrm{Tr} R \wedge \tilde{R}$ are such that

$$\int_X \alpha \mathrm{Tr} R^2 + \beta \mathrm{Tr} R\tilde{R} = \left(\frac{\chi + \sigma}{4} \right)$$

where χ and σ denote Euler characteristics and signature of the manifold X respectively.

The passage from (2.6) to the $\hat{A}$-genus is very similar to the evaluation of the one-loop diagrams. It leads to the following expressions for the functions $\mathcal{F}$ and $\mathcal{P}$ entering (2.7) (we write the value of the functionals on the $\phi = a \in \underline{\mathbf{t}}$ and we introduce R to keep track of dimensions):

$$\begin{aligned} \mathcal{F}(a) + \mathcal{F}^{\mathrm{pert}}(a)_{\mathrm{pureYM}} \quad &= \quad \frac{i}{8\pi^3 R^2} \sum_{\alpha \in \Delta} \mathrm{Li}_3\left(e^{2\pi R \langle a, \alpha \rangle}\right) \\ \mathcal{P}(a) \quad &= \quad \log \prod_{\alpha > 0} \frac{\pi R \langle a, \alpha \rangle}{\sinh\left(\pi R \langle a, \alpha \rangle\right)} \end{aligned} \tag{2.8}$$

(In other words, $\partial^2 \mathcal{F} / \partial x^2 \sim -\log(1 - e^{-x}) - \log(1 - e^{x}) + \log x^2$, for $x = \langle \alpha, 2\pi a \rangle$). In the formula (2.8) we use the expression for the perturbative prepotential of the pure $N = 2$ theory:

$$\mathcal{F}^{\mathrm{pert}}(a)_{\mathrm{pureYM}} = \sum_{\alpha \in \Delta} \frac{i \langle a, \alpha \rangle^2}{8\pi} \log \frac{\langle a, \alpha \rangle^2}{\Lambda^2}$$

For example, for $G = SU(2)$ the perturbatively exact coupling is given by the following expression:

$$\tau(a) = \frac{i}{2\pi} \log\left(\frac{1}{(\Lambda R)^2} \sinh^2\left(\frac{aR}{2}\right)\right) \tag{2.9}$$

One easily recognizes the presence of the group-like Vandermonde determinant in this formula (compare with the Lie-algebraic one in the perturbative four-dimensional formulae).

2.2. Non-perturbative answer

2.2.1.The case $G = SU(2)$

Recall the answer for the pure four dimensional theory, as presented in [2]. Consider the periodic Toda chain (for $G = SU(2)$)

$$h(p, q) = \frac{p^2}{2} + \Lambda^2 \cosh(q)$$

There is a family of elliptic curves E_u:

$$u = h(p, q)$$

associated with this system. Here p and q are allowed to vary in the complex region. The claim of [1],[2] is that the modular parameter $\tau(u)$ of the curve E_u in the family coincides with the gauge coupling in the low-energy abelian theory. The fact that $\tau(u)$ is defined up to the modular transformation corresponds to the electric-magnetic duality phenomenon in the abelian gauge theory.

In our case the low-energy four dimensional theory exhibits the electric-magnetic duality as well (in the purely five-dimensional theory the vector multiplet gets mapped to the tensor multiplet under the duality transformation, but upon compactification on the circle the tensor multiplet becomes equivalent to the vector multiplet as well) so it is natural to look for a family of elliptic curves encoding the answer for the non-perturbative prepotential.

We claim that this family is a relativistic generalization [4]of the Toda chain:

$$H(p, q) = \frac{1}{R^2} \cosh(Rp) \sqrt{1 + 2(\Lambda R)^2 \cosh(q)}$$

In the limit $\Lambda R \to 0$ this Hamiltonian reduces to the standard one (in the notations of Ruijsenaars $1/R$ corresponds to the speed of light).

The family of the curves $\mathcal{E}_U$ defined as $U = H(p, q)$ yields the correct asymptotics of the coupling constant $\tau(U)$. It is instructive to study the periods themselves ($\zeta = \Lambda R$):

$$\begin{aligned} \vec{A} &= \oint_{\vec{\gamma}} p dq \\ \frac{\partial \vec{A}}{\partial U} &= \oint_{\vec{\gamma}} \frac{R dq}{\sqrt{R^4 U^2 - 1 - 2\zeta^2 \cosh(q)}} \end{aligned} \tag{2.10}$$

Using

$$U = \frac{1}{R^2}\cosh\alpha, \qquad \nu_5 = 2\frac{\zeta^2}{\sinh^2\alpha}$$

(2.10) can be rewritten as:

$$\frac{\partial\vec{A}}{\partial\alpha} = \frac{1}{R}\oint_{\vec{\gamma}} \frac{dq}{\sqrt{1-\nu_5\cosh(q)}} \tag{2.11}$$

Let us compare (2.10) with the periods $\vec{a}$ of the pure four dimensional theory, where for:

$$u = \frac{\aleph^2}{2}, \qquad \nu_4 = \frac{\Lambda^2}{u}$$

one has:

$$\frac{\partial\vec{a}}{\partial\aleph} = \frac{1}{2\pi\Lambda}\oint_{\vec{\gamma}} \frac{dq}{\sqrt{1-\nu_4\cosh(q)}} \tag{2.12}$$

This implies that the curves E_u and $\mathcal{E}_U$ are isomorphic for $\nu_4 = \nu_5$, i.e. for

$$u = R^2U^2 - \frac{1}{R^2}$$

In particular the asymptotic $\tau(u) \sim \log(\frac{u}{\Lambda^2})$ of the curve E_u gets mapped to $\tau(U) \sim \log(\frac{R^4U^2-1}{R^2\Lambda^2})$ for the curve $\mathcal{E}_U$. Together with the asymptotic $U \sim \frac{1}{R^2}\cosh(2\pi AR)$ this yields (2.9).

The elliptic curve E_u degenerates for $\nu_4 = \pm1, \infty$. This corresponds to

$$U = \pm\frac{1}{R^2}\sqrt{1\pm\zeta^2}, \infty$$

Notice that in the limit $R \to \infty$ all singular points different from $U = \infty$ collide at $U = 0$, which corresponds to the restoration of the non-abelian gauge symmetry. This phenomenon has been noticed by several authors (e.g. [5], [10]). Also notice that we have five singular points as opposed to the notorious three points of the four-dimensional theory. This relies on the fact that in the absence of the fields in the fundamental representation the low-energy theory has a $\mathbb{Z}_2$ symmetry, which maps U to $-U$. This symmetry has the following origin. The order parameter U is nothing but the trace in the fundamental representation of the "Wilson loop" $g = P\exp\oint(A_t + i\varphi)$. If all fields under consideration are in the adjoint representation of the group G, then the multiplication of the Wilson loop g by the element of the center of the gauge group does not change physics. Therefore the true moduli space of the theory is the quotient of the U plane by the symmetry $U \to -U$. This symmetry is broken as long as one has a fundamental matter.

2.2.2.The general case: $G = SU(N)$

In the rank $N-1$ case the integrable system behind the four dimensional theory is known to be the periodic Toda chain. It is an integrable system on the $2(N-1)$ complex dimensional phase space with the canonical coordinates (p_i, q_i), $i = 1, \ldots, N$ which obey the constraint:

$$\sum_i q_i = \sum_i p_i = 0$$

As any integrable system is has $N-1$ integral in involution, i.e. the set of functions $I_2, \ldots, I_N$ of p and q such that they Poisson-commute and are functionally independent. The simplest way to see these integrals of motion is through the Lax operator. It is a $N \times N$ matrix L, whose entries depend on p, q and also on the auxiliary parameter z:

$$\begin{aligned} L_{ij}(z) &= p_i \delta_{ij} + \delta_{i,j-1} + \delta_{i,j+1} \Lambda^2 e^{q_i - q_{i-1}} \\ &- \Lambda^N z \delta_{i,N} \delta_{j,1} + \Lambda^{2-N} z^{-1} e^{q_1 - q_N} \delta_{i,1} \delta_{j,N} \end{aligned} \tag{2.13}$$

By definition, the generating function of the integrals of motion of the Toda lattice is the characteristic polynomial

$$\det(L(z) - w) = \sum_{k=0}^{N} w^k I_k + (-\Lambda)^N z + \Lambda^N z^{-1} \tag{2.14}$$

The action-angle variables of the system play the vital role in the solution of the supersymmetric gauge theory. Namely, the action variables become the central charges of the susy algebra a^i which encode the behavior of couplings τ_{ij}. The angle variables are hidden in the four-dimensional story but become visible (as moduli) as long as one compactifies one dimension – they become the vacuum expectation values of the fourth component of the gauge field and of the scalar dual to the three dimensional photon (see [13]).

The generalization of this story to the theory on $\mathbb{R}^4 \times \mathbf{S}^1$ makes use of the Lax operator for the relativistic periodic Toda chain (see, for example [4], [14]):

$$\begin{aligned} L_{ij} &= e^{Rp_i} f_i (l_{ij} + b_{ij}) \\ l_{ij} &= \delta_{i,j+1}(1 + \zeta^N z)\xi_i - \delta_{i,1}\delta_{j,N}(1 + \zeta^{-N} z^{-1})\xi_1 \\ b_{ij} &= \begin{bmatrix} -(i\zeta)^N, & i \leq j-1 \\ \\ 1, & i > j-1 \end{bmatrix} \\ f_i^2 &= (1 - \zeta^2 e^{q_{i+1} - q_i})(1 - \zeta^2 e^{q_i - q_{i-1}}) \\ \xi_i^{-1} &= 1 - \zeta^{-2} e^{q_{i-1} - q_i} \end{aligned} \tag{2.15}$$

with $q_{N+1} = q_1, q_0 = q_{N-1}$.

The spectral curve $\det(L(z) - w) = 0$ can be constructed using the results of [4]. One has:

$$\begin{aligned} \det(L(z) - w) &= \sum_{l=0}^{N} (-w)^{N-l} \sigma_l \\ &= (-w)^N + \sum_{l=1}^{N-1} (1 + \zeta^N z)^{l-1} (-w)^{N-l} S_l \\ &\quad + (1 + \zeta^N z)^{N-1} (1 + \zeta^N z^{-1}) \end{aligned} \tag{2.16}$$

where σ_l is the l'th symmetric function of $L(z)$ and S_l are the z-independent coefficients. After the change of variables

$$y = -\frac{1 + \zeta^N z}{w}, \quad \chi = z y^{-N/2} \tag{2.17}$$

the curve acquires the form

$$y^{-N/2} \mathcal{P}(y) = \zeta^N (\chi + \chi^{-1}) \tag{2.18}$$

with $\mathcal{P}$ denoting the N'th degree polynomial (which starts with 1 and ends with y^N) and the differential is

$$\lambda = \frac{1}{2\pi i} \log(y) \frac{d\chi}{\chi}$$

The proposal passes through a number of checks. The first one is the comparison to the known solutions of the $N = 2$ super-Yang-Mills theory with the gauge group $G = SU(N)$ [15]. The solution makes use of the family of hyperelliptic curves

$$y^2 = P(x)^2 - \Lambda^{2N} \tag{2.19}$$

where $P(x) = \det(x - \phi)$ is the characteristic polynomial of the Higgs field. In the Coulomb phase the Higgs field can be diagonalized: $\phi = \mathrm{diag}(a_1, \ldots, a_N)$.

$$\det(\phi - x) = \prod_i (a_i - x)$$

The five-dimensional gauge theory with the gauge group G can be understood as a four dimensional theory with the gauge group $\hat{G}$ - the extended loop group. The analogue of (2.19)for $\hat{G}$ is easy to find: replace the Higgs field $\phi(x)$ by the first order differential operator on the circle:

$$\phi \to \partial_t + A_t + i\varphi$$

The determinant $\det(\partial_t + A_t + i\varphi - x)$ can be conveniently defined using the zeta-regularization with the result:

$$\det(\partial_t + A_t + i\varphi - x) = \prod_i \frac{1}{R} \sinh(2\pi R a_i - x)$$

Therefore the curve [15] assumes the form:

$$y^2 = \prod_i \frac{1}{R^2} \sinh^2(2\pi R a_i - x) - \Lambda^{2N} \tag{2.20}$$

The convenient curve [2]

$$z + \frac{\Lambda^{2N}}{z} = P(x) \tag{2.21}$$

goes over to

$$z + \frac{\Lambda^{2N}}{z} = \prod_i \left(\frac{\sinh(2\pi R a_i - x)}{R} \right) \tag{2.22}$$

Introduce the variables $y = e^{2x}$, $\chi = \frac{z}{\Lambda^N}$. Then, thanks to $\sum_i a_i = 0$ we rewrite (2.22)as:

$$\zeta^N \big(\chi + \frac{1}{\chi} \big) = y^{-N/2} \mathcal{P}_N(y) \tag{2.23}$$

with

$$\mathcal{P}_N(y) = 1 + I_1 y + \ldots + I_{N-1} y^{N-1} + y^N \tag{2.24}$$

which coincides with the spectral curve (2.18)of the relativistic periodic Toda system. We observe the $\mathbb{Z}_N$ degeneracy of the five-dimensional moduli space compared to the four-dimensional one : under the transformations

$$y \to e^{\frac{2\pi i k}{N}} y, \quad I_l \to e^{-\frac{2\pi i k l}{N}} I_l, \quad z \to (-)^k z \tag{2.25}$$

the curve and the differentials $\partial \lambda / \partial I_l$ are not changed. These transformations correspond to the multiplications of the Wilson loop $P \exp \oint (A_t + i\varphi)$ by the elements of the center of $SU(N)$.

2.3. $N_f > 0$

The case with the fundamental matter can, in principle, be treated with the help of integrable systems, although we were not able to identify them (see [16], though). Instead for the sake of brevity we present the family of curves with the differentials using the remarks at the end of the previous section.

The family of curves describing the $N = 2$ theory with N_f hypermulitplets in the fundamental representation of $G = SU(N_c)$ gauge group (see also [17]for the more general case) is known.

$$
\begin{aligned}
z + \frac{\Lambda^{2N_c - N_f} P_{N_f}(x)}{z} =& P_{N_c}(x) \\
P_{N_f}(x) =& \prod_{k=1}^{N_f} (x + m_k) \\
P_{N_c}(x) =& \det(x - \phi)
\end{aligned}
\tag{2.26}
$$

In five dimensions the mass of a hypermultiplet is a real scalar which can be thought of the scalar component of the vector multiplet. Upon compactification on the circle of the radius R it becomes a complex scalar, whose imaginary part lives on the circle of the radius $\frac{1}{R}$, i.e. the physics must be invariant under $m_i \to m_i + \frac{i n_i}{R}$ for $n_i \in \mathbb{Z}$. This suggests the replacements:

$$
\begin{aligned}
P_{N_f}(x) =& \prod_{k=1}^{N_f} \frac{1}{R} \sinh(x + 2\pi R m_k) \\
P_{N_c}(x) =& \prod_{i=1}^{N_c} \frac{1}{R} \sinh(x - 2\pi R a_i)
\end{aligned}
\tag{2.27}
$$

Introducing once again $y = e^{2x}$ we write the new family of curves as:

$$
z + \frac{\mathcal{P}_{N_f}(y)}{z y^{N_f/2}} = y^{-N_c/2} \mathcal{P}_{N_c}(y) \tag{2.28}
$$

with $\mathcal{P}_{\ldots}$ again denote the relevant polynomials.

2.4. Adjoint matter

The theory with the massive adjoint hypermultiplet is peculiar in the following sense: it is ultra-violet finite in four-dimensions and seems to be ultra-violet finite in five dimensions too. The perturbative coupling behaves like (m is the mass of the hypermultiplet, $G = SU(2)$):

$$
\tau^{eff} \sim \tau^{bare} + \frac{i}{2\pi} \log \frac{\sinh^2(Ra)}{\sinh(Ra + 2\pi R m) \sinh(Ra - 2\pi R m)} \tag{2.29}
$$

and therefore stops running at sufficiently large a. Assuming this finiteness we introduce the microscopic coupling τ (the issue of the relation of the five dimensional real coupling and the complex τ is discussed below). Now we look for the integrable system, which has to be defined with the help of the elliptic curve E_τ, reduces at $R = 0$ to the elliptic Calogero-Moser system (as it describes the four dimensional story [3]) and also reduces to the relativistic Toda system in when hypermultiplet becomes heavy enough.

Fortunately, the collection of relativistic integrable systems contains the one which is the natural candidate for the theory with adjoint hypermultiplet. It is the elliptic Ruijsenaars-Schneider model (or Relativistic Calogero-Moser system) [18].

It has the Lax operator:

$$L_{ij} = e^{Rp_i} f_i(q) \frac{\sigma(q_{ij} + z)\sigma(iRm)}{\sigma(q_{ij} + iRm)\sigma(z)} \tag{2.30}$$

where

$$f_i(q)^2 = \prod_{k \neq i} (1 - \frac{\wp(q_{ik})}{\wp(iRm)}), \tag{2.31}$$

$q_{ij} = q_i - q_j$, σ is the Weierstrass sigma function and $\wp = -(\log \sigma)''$. For example, the Hamiltonian which replaces

$$H^{non-rel} = \sum_i \frac{1}{2} p_i^2 + m^2 \sum_{i \neq j} \wp(q_{ij})$$

has the form

$$H^{rel} = \frac{1}{R^2} \sum_i \cosh(Rp_i) f_i(q)$$

Let us show that the parameter m in the formulae (2.30), (2.31) indeed corresponds to the mass of the hypermultiplet. More precisely, we shall show that the symplectic form $\sum_i dp_i \wedge dq^i$ has a non-trivial period, proportional to m (this makes contact with the ideology of [1]).8 To simplify things we restrict ourselves with $G = SU(2)$ case and consider the spectral curve

$$U = \cosh(Rp) \sqrt{1 - \frac{\wp(q)}{\wp(iRm)}} \tag{2.32}$$

8 This statement is local in the m space. As one goes around the cycle m gets shifted and so does the period. This has to do with the monodromy representation of the π_1 of m space in $H_2(\mathcal{M}, \mathbb{Z})$.

The symplectic form $dp \wedge dq$ can be rewritten as:

$$\omega = dp \wedge dq = \frac{dy \wedge dq}{2R\sqrt{(y - \wp(iRm))(y - \wp(q))}} \tag{2.33}$$

where $y = \wp(iRm) + \cosh^2(Rp)(\wp(q) - \wp(iRm))$. Now we describe a non-contractible two-cycle Γ and compute the integral

$$\int_\Gamma \omega$$

In the y-plane for the fixed q there is a cut between the points $\wp(q)$ and $\wp(iRm)$. There is a one-cycle C_q which goes around the cut. The period

$$\oint_{C_q} \alpha(q)$$

of the one-differential

$$\alpha(q) = \frac{dy}{\sqrt{(y - \wp(iRm))(y - \wp(q))}}$$

equals $2\pi i$. The cycle C_q vanishes when $\wp(q) = \wp(iRm)$. Therefore there is a two-cycle Γ (a two-sphere) whose projection onto the elliptic curve E_τ (where q lives) is the contour going from $q = -iRm$ to $q = +iRm$ and the fiber over a point q is C_q. Clearly,

$$\frac{1}{2\pi i}\int_\Gamma \omega = \frac{1}{2R}\int_{-iRm}^{+iRm} dq = im \quad \big(+\frac{1}{2R}(n_1 + n_2\tau)\big) \tag{2.34}$$

(we see that the periods are half-integral. This has to do with the center of $SU(2)$)

One can proceed similarly in the general case $N > 2$. Notice that for the two degenerations of the elliptic Ruijsenaars-Schneider model, namely, for the trigonometric limit ($\tau \to \infty$) and for the non-relativistic limit ($R \to 0$) the linear dependence of the cohomology class of the symplectic form ω follows from a holomorphic variant of the Duistermaat-Heckman theorem, as in those cases there is a description of the models via the Hamiltonian reduction [19], [20]. Recently the elliptic Ruijsenaars model has been given the Hamiltonian reduction description [21].

<u>A bizarre symmetry</u>. Notice that the solution of the model with the adjoint hypermultiplet has the following symmetry:

$$m \to m + \frac{i}{2R}(n_1 + n_2\tau), \qquad n_{1,2} \in \mathbb{Z} \tag{2.35}$$

The first symmetry (n_1) reflects the presence of the tower of Kaluza-Klein states in the four dimensional theory. To understand the second one we have to recall the relation between

the couplings in four and five dimensions. In five dimensions $\frac{1}{g_5^2}$ has the dimension of the mass and enters the vector multiplet whose vector component gauges the topological current $\mathrm{Tr} F \wedge F$. Again, upon compactification down to four dimensions on a circle it becomes a complex scalar τ which is conventionally written as:

$$\tau = \frac{4\pi i R}{g_5^2} + \frac{\theta}{2\pi}$$

where θ is the integral of the corresponding gauge field along the circle $\mathbf{S^1}$. So, if the bare coupling in five dimensions is finite then the theory must have a symmetry:

$$m \to m + \frac{n}{g_5^2}, \quad n \in \mathbb{Z}$$

already in the decompactification limit. We suspect that the presence of the tower of states here has to do with the tensionless strings in six dimensions. Upon the compactification on a small circle the six dimensional string gives rise to the tower of particles in five dimensions. Indeed it was suggested by E. Witten in [22] that the S-duality of $N = 4$ SYM in four dimensions is best understood as the consequence of the existence of the self-dual non-critical string in six dimensions. In five dimensions the theory with the massless adjoint hypermultiplet gives rise to $N = 4$ SYM in four dimensions. The moduli space seems to be locally flat (as it is in four dimensions), the only difference being that the scalar a lives on a circle. In other words, the vector moduli space is $\mathbf{T}_{\mathbb{C}}/W$, where $\mathbf{T}_{\mathbb{C}}$ is the complexification of the Cartan subgroup of the gauge group G.

On the other hand, if we are to have a finite coupling in the limit $R \to 0$ then g_5^2 better be of the order of R and therefore the symmetry (2.35)becomes vacuous.

The proposed solution identifies the periods of the differential $\lambda = pdz$ (the action variables) with the susy central charges. The subtlety here is that the spectral curve for $N > 2$ has a genus $N(N-1)/2$ as opposed to the non-relativistic case. The problem of identification of the relevant periods has been recently solved [23].

<u>Flow to the pure gauge theory</u>. One expects to recover the minimal theory in the limit where the adjoint hypermultiplet becomes very heavy. It is clear, though, that the microscopic coupling τ must be tuned in such a way that the limit $m \to \infty$ becomes possible.

Historically, the relativistic Calogero-Moser model was discovered before the corresponding Toda system [18]. The trick producing the periodic Toda system out of the

elliptic Calogero-Moser system is precisely the flow we need. It is completely analogous to the non-relativistic case. One shifts the q_i variables as:

$$q_j \to q_j - \frac{j}{N}\tau$$

After that one takes $\tau \to i\infty$, keeping

$$\Lambda R = \exp\bigl(2\pi R\bar{m} + 2\pi i\frac{\tau}{N}\bigr) \tag{2.36}$$

(with $\bar{m} = m \cdot \mathrm{sgn}(\Re m)$) finite [4]. This renormalization is also suggested by (2.29). One has to be careful comparing (2.36) to the $R = 0$ case [3], since it corresponds to the hypermultiplet which is much heavier then the Kaluza-Klein modes ($m >> \frac{1}{R}$) and therefore the RG flow looks different from the conventional one (which corresponds to $\frac{1}{R} >> m >> a$). Given the existence of all four integrable systems (Toda, Calogero-Moser and their relativistic analogues) together with their limiting properties one can reconstruct the whole picture of flows. We plan to return to this issue elsewhere.

3. Future directions and conclusions

Among the immediate future generalizations one can recognize the solution of the six-dimensional gauge theory. The corresponding Hamiltonians are elliptic functions on the rapidities p (this is in the slight contradiction with the statements of [24]). The corresponding integrable model has not been known before and can be constructed along the lines of [25], as dual to the known elliptic systems. The perturbative prepotential for this model has been computed in [12]. It corresponds to the elliptic genus of the moduli space of instantons (which is a generalization of the $\hat{A}$ -genus). An interesting step towards the actual computation of the elliptic genera has been made in [26]. The compactifications of the six-dimensional theory has been recently studied in a several contexts. In particular, the sums of the trilogarithms appeared in [8]. The low-energy effective action for the six-dimensional tensionless string compactified on a two-torus was studied in [27].

It is also interesting to compare our results with those of [13]. Consider the five dimensional theory compactified on a two-torus down to three dimensions. The moduli space is now $4r$ dimensional for $r = \mathrm{rk}G$ and has a hyperkähler structure. This structure is parameterized by the geometry of the two-torus. For example for a rectangular torus it depends on two radii R_4, R_5 and in the limit when one of the radii goes to zero we get

the Atiyah-Hitchin's monopole space $\widetilde{\mathcal{M}}_2^0$ [13]. In the opposite case where one of the radii goes to infinity we expect to see the relativistic integrable system.

It is even more interesting to start with the six dimensional theory. Let us imagine that we have an $SU(2)$ (what are we saying is possible to extend for a generic gauge group G, but we restrict ourselves with $G = SU(2)$. This is also relevant to M-theory compactification on $K3 \times \mathbb{R}^7$) $N = 1$ supersymmetric gauge theory on $\mathbf{T}^3 \times \mathbb{R}^3$. We take the metric $G_{ab} d\varphi^a d\varphi^b$ on $\mathbf{T}^3$ to be flat to preserve supersymmetry.

The classical moduli space $\mathcal{M}$ of the effective three dimensional theory is a copy of an orbifold limit of $K3$ manifold (this has been recently noticed by the several authors [28], [29]). Indeed, the scalars in the effective theory are the Wilson loops around three cycles in $\mathbf{T}^3$ and the scalar, dual to the three dimensional photon. The Wilson loops are the commuting elements of G and can be simultaneously diagonalized, i.e. they define three elements of $U(1)$. They are defined uniquely up to the transformation in the Weyl group, which also acts on the scalar σ, dual to the photon. The scalar σ lives actually on the circle of the circumference 2π. Thus we get the manifold of classical vacua, isomorphic to $\mathbf{T}^4/\mathbb{Z}_2$.

The classical metric on the moduli space is flat with the orbifold singularities at the fixed points of $\mathbb{Z}_2$ action. It seems that the quantum corrections make the metric smooth at this points. The $N = 4$ $3d$ supersymmetry guarantees that it should be hyper-Kähler.

Our solutions probe the different regions in the $K3$ moduli space, where (for four- and five-dimensional theories) $K3$ looks non-compact in a number of directions.

Our results in the limit $R \to \infty$ reproduce the formulas of [5]. It would be interesting to find the prepotentials (as the functions of R) for other examples considered in [10], [9],[30], [31].

Our results will be used in the derivation of the four dimensional analogues of Verlinde formulas [32]They count the number of holomorphic blocks in the theories, which were constructed and studied in [33],[34], [12], [35].

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5. Appendix. Supersymmetric Quantum Mechanics on the Moduli Space

In this section we discuss the five-dimensional theory on a manifold $X \times S^1$ in the limit where X is effectively very small. The theory reduces to the quantum mechanics whose target space is the moduli space of instantons on X.

Recall that the action of the theory has the form

$$
\begin{gathered}
S = \int_{X \times \mathbf{S}^1} \theta \wedge \mathrm{Tr}(F \wedge F) + \{Q, \alpha_m R_m + \alpha_a R_a + \alpha_k R_k\} \\
R_a = \eta D_t \varphi \qquad\qquad R_k = \psi^\mu (F_{\mu t} - i D_\mu \varphi) \\
R_m = i\chi^{\mu\nu}(F^+_{\mu\nu} - e_0^2 H^+_{\mu\nu})
\end{gathered}
\tag{5.1}
$$

where θ is the background gauge field (it is in the vector multiplet, whose scalar component is nothing but the five dimensional bare coupling $\frac{1}{e_0^2}$).

If all the observables are Q-closed we are free to vary the parameters $\alpha_m, \alpha_k, \alpha_a$ as long as appropriate non-degeneracy conditions are fulfilled. The answer will remain intact.

Weak coupling limit. By the weak coupling we understand here the limit where $\alpha_m \to \infty$, $e_0^2 \to 0$. We also can assume that $\alpha_a \to 0$. As α_m multiplies $H^{\mu\nu} F^+_{\mu\nu}$ the limit $\alpha_m \to \infty$ constrains the space9-like part A_μ of the gauge field to be anti-self-dual:

$$
F^+_{\mu\nu} = 0 \qquad (D_\mu \psi_\nu)^+ = 0
$$

The moduli space $\mathcal{M}$ of solutions to this equation up to the gauge transformations is finite-dimensional (for a given topological sector). Let $m^I, I = 1, \ldots, \mathrm{dim}\mathcal{M}$ be some choice of the local coordinates on $\mathcal{M}$. The path integral becomes an integral over the space of loops in $\mathcal{M}$. To write an effective Lagrangian for the quantum mechanics we need a description of the geometry of the moduli space (the description is similar to the one presented in [36], [37]).

Let us consider a particular solution A (a connection on a vector bundle E) of the equation $F^+ = 0$ and take its deformation $A' = A + a$. It satisfies the linearized equations $D^+_A a = 0$. The trivial solutions to this equation $a = D_A \epsilon$ are the infinitesimal gauge transformations. The true tangent space $T_A\mathcal{M}$ is the first cohomology group $H^1 = \mathrm{Ker} D^+_A / \mathrm{Im} D_A$ of the Atiyah-Hitchin-Singer complex:

$$
0 \;\to\; \Omega^0(X, adE) \;\xrightarrow{D_A}\; \Omega^1(X, adE) \;\xrightarrow{D^+_A}\; \Omega^{2,+}(X, adE) \;\to\; 0
\tag{5.2}
$$

9 We refer to X as to the space

For the moduli space to be smooth one needs other cohomology groups to vanish. More generally, the compactified moduli space is a stratified space (stack) whose strata have constant dimensions of all cohomology groups of (5.2). This allows to construct the (formal) characteristic classes of the tangent bundle, defined as an element of the K-functor of $\mathcal{M}$. Now assume that the virtual dimension of $\mathcal{M}$ (which is the index of the complex (5.2)) coincides with the dimension of H^1. We can choose a basis in this space and represent it by the solutions to the linearized instanton equations, which obey some gauge fixing condition. A convenient choice of gauge is $D_A^* a = 0$. So, let us pick a set of linearly independent solutions to the equations:

$$D_A^+ a_I = 0 \qquad D_A^* a_I = 0 \tag{5.3}$$

for $I = 1 \ldots, \mathrm{dim}\mathcal{M}$. The metric on the moduli space has the form:

$$G_{IJ} = \int_X a_I \wedge * a_J \tag{5.4}$$

Now let us consider a gauge field, whose space-like part $A_\mu(x;t)dx^\mu$ solves the instanton equations. There exists (locally in $\mathcal{M}$) a section (a universal instanton) $\mathcal{A}(m) = \mathcal{A}_\mu dx^\mu(x;m)$. We have

$$\begin{aligned} \partial_t A_\mu(x;t) =& \frac{\partial m^I}{\partial t}\frac{\partial \mathcal{A}_\mu}{\partial m^I} \\ \frac{\partial \mathcal{A}}{\partial m^I} =& a_I + D_A \epsilon_I \end{aligned} \tag{5.5}$$

where ϵ_I is the compensating gauge transformation ($\mathcal{A} + \epsilon$ define a connection in the universal bundle $\mathcal{E}$ over $\mathcal{M} \times X$).

Now let us have a look at the remaining part of the action:

$$\{Q, R_k\} = F_{\mu t}^2 + (D_\mu \varphi)^2 + \psi^\mu D_\mu \eta + \psi^\mu [D_t - i\varphi, \psi_\mu]$$

The fermion $\psi = \psi_\mu dx^\mu$ already satisfies $D_A^+ \psi = 0$ (after integrating out χ) and the integral over η enforces the condition $D_A^* \psi = 0$. Thus, $\psi = \chi^I a_I$, where χ^I are the fermionic coordinates, tangent to $\mathcal{M}$. Introduce the covariant derivative

$$\nabla_I = \frac{\partial}{\partial m^I} + \epsilon_I$$

We have:

$$[\nabla_I, D_A] = a_I$$

Clearly, $d_{\mathcal{M}} \sim \chi^I \nabla_I$. We also have: $F_{\mu t} = a_I \dot{m}^I + D_\mu(\epsilon_I \dot{m}^I - A_t)$. Introduce the 'improved' connection

$$B = A_t - \epsilon_I \dot{m}^I + i\varphi$$

Substituting the expressions for all fields in the action we get:

$$S = \int_{S^1} G_{IJ}[\dot{m}^I \dot{m}^J + \chi^I \nabla_t \chi^J] + \int_X d^4x \sqrt{g} D_\mu B D^\mu \bar{B} + \psi_\mu [\bar{B}, \psi^\mu] \tag{5.6}$$

After integrating out $\bar{B}$ we are left with the standard action of the $N = \frac{1}{2}$ supersymmetric quantum mechanics with $\mathcal{M}$ as a target space. We also have:

$$B = \frac{1}{\Delta}[\psi, *\psi] = \Phi_{IJ} \chi^I \chi^J$$

It is important to notice that Φ_{IJ} has a local form, in fact it is a curvature two-form of the connection ∇_I (this discussion is parallel to the one in [37]) :

$$\Phi_{IJ} = [\nabla_I, \nabla_J]$$

as follows from the identities:

$$a_I = [\nabla_I, D_A] \Rightarrow [a_I, *a_J] = D_A^* D_A \Phi_{IJ}$$

The main conclusion is that we have reduced the path integral in the five-dimensional theory to the one of the supersymmetric quantum mechanics on $\mathcal{M}$. Using the standard localization arguments one reduces the partition function to the integral over $\mathcal{M}$ of the $\hat{A}$-genus density:

$$\hat{A}(\mathcal{M}) = \det \frac{\frac{\mathcal{R}}{4\pi i}}{\sinh \frac{\mathcal{R}}{4\pi i}}$$

<u>Strong coupling limit.</u> In the opposite limit $\alpha_m \to 0$, $\alpha_a \to \infty$ (which corresponds to the low energy expansion of the effective four-dimensional theory) one has the potential $(D_t \varphi)^2$ which breaks the gauge group G down to its maximal torus T (quite similarly to the conventional four dimensional story). The subtlety is that the order parameter has the periodicity property. Indeed, one can find a gauge

$$A_t + i\varphi = \sum_i a_i H_i$$

with $\partial_t a_i = 0$, H_i being the generators of the Cartan subalgebra $\underline{\mathbf{t}}$. This gauge leaves out the discrete gauge transformations, which act on a_i as follows:

$$a_i \to a_i + 2\pi i \frac{n_i}{R}, \quad n_i \in \mathbb{Z}$$

Eventually we get the four-dimensional effective theory, described in the section 2.1, where the periodicity gets restored once the one-loop prepotential coming from the massless photon(s) is taken into account.

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